Isoparametric formulation of strain based rectangular finite elements

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Abstract

Two finite elements based on the strain approach have been reformulated using the isoparametric formulation procedure. This isoparametric procedure presented in this work has been used for the first time in the case of these strain based elements in order to establish the element stiffness matrix numerically instead of having it analytically as usual. This will allow having the extension of the strain approach to other analysis. The two elements are membrane rectangular with four nodes. The first has the two essential translations at each of the four corner nodes whereas the second has in addition a drilling rotation. Performance of these elements is evaluated through a series of tests case, and the obtained results confirm their precision in linear and nonlinear analysis.

Key words: strain approach, finite element, membrane element, *isoparametric formulation*, *linear analysis*.

1. Introduction

Many researchers have adopted for a long time the strain based approach for the development of new finite elements. The essential goal was especially related to the performance of these elements. The first developed elements were only concerned with curved ones [1, 2]. This approach was later extended to plane elasticity elements [3-5], for three-dimensional elasticity [6], for cylindrical shells [7-10], and for plate bending [11].

The ability to solve linear and nonlinear problems is more important in many aspects of finite element work. In fact, exact solutions for both problems only exist for a few simple cases, so the use of the finite element method is required. However the use of the classical isoparametric displacement-based elements becomes increasingly inefficient and leads to a considerable gain on computing times for this type of analysis. The advantages of the strain based finite elements have been illustrated on several elements [12, 13] compared with displacement-based ones. However the element stiffness matrix of the strain based elements is usually obtained by analytical integration. Then it turned out that the isoparametric reformulation of these elements is necessary in order to make their extension to other analysis

In this context two finite elements based on the strain approach named SBRIE [14] (**S**train **B**ased **R**ectangular **I**n Plane **E**lement) and SBRIEIR [15] (**S**train **B**ased **R**ectangular **I**n Plane **E**lement with **I**n plane **R**otation) have been reformulated using the isoparametric procedure and are used in the elastic and elasto-plastic analysis. For the purpose of demonstration, some selected numerical examples are solved using these two elements, and the obtained results confirm their good precision in linear and nonlinear analysis.

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2. Isoparametric formulation

2.1. SBRIE element

The rectangular element SBRIE is schematically shown in Figure 1. The lengths a and b are respectively the dimensions of the element in x and y direction.

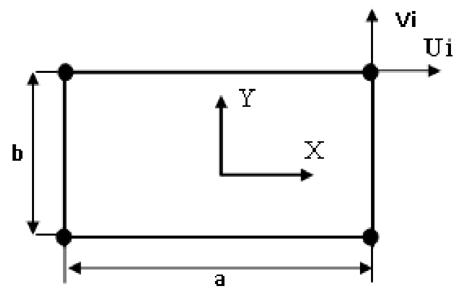

Figure 1. Coordinates and nodal point for the SBRIE element

The assumed strain functions and the corresponding displacement fields of this element in Cartesian coordinates are given by

$$\begin{aligned} \varepsilon_x &= a_4 + a_5 y \\ \varepsilon_y &= a_6 + a_7 x \\ \gamma_{xy} &= a_8 \end{aligned} \tag{1}$$

$$\begin{aligned} u &= a_1 - a_3 y + a_4 x + a_5 xy - a_7 y^2/2 + a_8 y/2 \\ v &= a_2 + a_3 x - a_5 x^2/2 + a_6 y + a_7 xy + a_8 x/2 \end{aligned} \tag{2}$$

The displacement field is given in term of the nodal displacement as follows

$$\{U\} = [N][q] \tag{3}$$

Where:

$$\{q\} = \{u_1 \ \mathrm{v}_1 \ \mathrm{u}_2 \ \mathrm{v}_2 \ \mathrm{u}_3 \ \mathrm{v}_3 \ \mathrm{u}_4 \ \mathrm{v}_4\}^T \tag{4}$$

and the shape functions matrix [N] can be expressed as

$$[N] = [\varphi(x, y)][\phi_e]^{-1} \tag{5}$$

in which the matrix [$\varphi(x, y)$] and the matrix coordinates [ϕ_e] are respectively given by

$$[\varphi(x, y)] = \begin{bmatrix} 1 & 0 & -y & x & xy & 0 & -y^2/2 & y/2 \\ 0 & 1 & x & 0 & -x^2/2 & y & xy & x/2 \end{bmatrix} \tag{6}$$

$$[\phi_e] = \begin{bmatrix} \varphi(x_1, y_1) \\ \varphi(x_2, y_2) \\ \varphi(x_3, y_3) \\ \varphi(x_4, y_4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -y_1 & x_1 & x_1 y_1 & 0 & -0.5y_1^2 & 0.5y_1 \\ 0 & 1 & x_1 & 0 & -0.5x_1^2 & y_1 & x_1 y_1 & 0.5x_1 \\ 1 & 0 & -y_2 & x_2 & x_2 y_2 & 0 & -0.5y_2^2 & 0.5y_2 \\ 0 & 1 & x_2 & 0 & -0.5x_2^2 & y_2 & x_2 y_2 & 0.5x_2 \\ 1 & 0 & -y_3 & x_3 & x_3 y_3 & 0 & -0.5y_3^2 & 0.5y_3 \\ 0 & 1 & x_3 & 0 & -0.5x_3^2 & y_3 & x_3 y_3 & 0.5x_3 \\ 1 & 0 & -y_4 & x_4 & x_4 y_4 & 0 & -0.5y_4^2 & 0.5y_4 \\ 0 & 1 & x_4 & 0 & -0.5x_4^2 & y_4 & x_4 y_4 & 0.5x_4 \end{bmatrix} \tag{7}$$

Thus the assumed strain

$$\{\varepsilon\}=[Q(x,y)]\{a\} \tag{8}$$

With the strain matrix [Q] is given by

$$[Q(x,y)]=[L][\varphi(x,y)] \tag{9}$$

Where [L] is the differential operator matrix defined as

$$[L]=\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \tag{10}$$

Then the eq. (9) becomes

$$[Q(x,y)]=\begin{bmatrix} 0 & 0 & 0 & 1 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{11}$$

And the displacement field can be expressed as

$$\begin{Bmatrix} u \\ v \end{Bmatrix}=[\varphi(x,y)]\{a\} \tag{12}$$

In isoparametric formulation case the displacement interpolations can be written

$$\begin{aligned} u &= \sum_{i=1}^{n} N_{2i-1}^{U}(\xi,\eta)u_i + \sum_{i=1}^{n} N_{2i}^{U}(\xi,\eta)v_i \\ v &= \sum_{i=1}^{n} N_{2i-1}^{V}(\xi,\eta)u_i + \sum_{i=1}^{n} N_{2i}^{V}(\xi,\eta)v_i \end{aligned} \tag{13}$$

Then the shape functions [N] in eq. (5) can be written in the following form

$$[N]=\begin{bmatrix} N_1^u & N_2^u & N_3^u & N_4^u & N_5^u & N_6^u & N_7^u & N_8^u \\ N_1^v & N_2^v & N_3^v & N_4^v & N_5^v & N_6^v & N_7^v & N_8^v \end{bmatrix} \tag{14}$$

Where the obtained shape functions in natural coordinates are given in Table 1 and consequently the strain matrix [B] of SBRIE is given in table 2.

Table 1. Shape functions of SBRIE element

$N_1^u=1/4(1-\xi)(1-\eta)$	$N_1^v=1/8(1-\xi^2)$
$N_2^u=1/8(1-\eta^2)$	$N_2^v=1/4(1-\xi)(1-\eta)$
$N_3^u=1/4(1-\xi)(1+\eta)$	$N_3^v=1/8(\xi^2-1))$
$N_4^u=1/8(\eta^2-1)$	$N_4^v=1/4(1-\xi)(1+\eta)$
$N_5^u=1/4(1+\xi)(1+\eta)$	$N_5^v=1/8(1-\xi^2)$
$N_6^u=1/8(1-\eta^2))$	$N_6^v=1/4(1+\xi)(1+\eta)$
$N_7^u=1/4(1+\xi)(1-\eta)$	$N_7^v=1/8(\xi^2-1)$
$N_8^u=1/8(\eta^2-1))$	$N_8^v=1/4(1+\xi)(1-\eta)$

Table 2. The components of the strain matrix [B]of the SBRIE

$B_{11} = N^u_{1,x}$	$B_{21} = N^v_{1,y}$	$B_{31} = N^u_{1,y} + N^v_{1,x}$
$B_{12} = N^u_{2,x}$	$B_{22} = N^v_{2,y}$	$B_{32} = N^u_{2,y} + N^v_{2,x}$
$B_{13} = N^u_{3,x}$	$B_{23} = N^v_{3,y}$	$B_{33} = N^u_{3,y} + N^v_{3,x}$
$B_{14} = N^u_{4,x}$	$B_{24} = N^v_{4,y}$	$B_{34} = N^u_{4,y} + N^v_{4,x}$
$B_{15} = N^u_{5,x}$	$B_{25} = N^v_{5,y}$	$B_{35} = N^u_{5,y} + N^v_{5,x}$
$B_{16} = N^u_{6,x}$	$B_{26} = N^v_{6,y}$	$B_{36} = N^u_{6,y} + N^v_{6,x}$
$B_{17} = N^u_{7,x}$	$B_{27} = N^v_{7,y}$	$B_{37} = N^u_{7,y} + N^v_{7,x}$
$B_{18} = N^u_{8,x}$	$B_{28} = N^v_{8,y}$	$B_{38} = N^u_{8,y} + N^v_{8,x}$

2.2. SBRIEIR element

The rectangular element SBRIEIR is schematically shown in Figure 2. The lengths a and b are respectively the dimensions of the element in x and y direction.

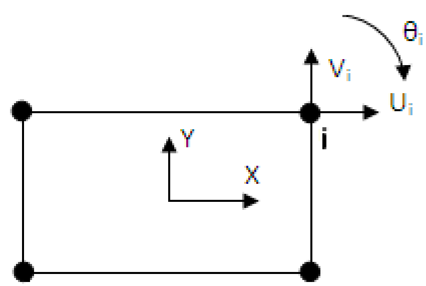

Figure 2. Co-ordinates and nodal point for the SBRIEIR element

The assumed strain functions and the corresponding displacement fields of this element in Cartesian coordinates are given by

$$\begin{aligned} \varepsilon_x &= a_4 + a_5 y + a_{11} y^2 + a_{12} x y^3 \\ \varepsilon_y &= a_6 + a_7 x - a_{11} x^2 - 2a_{12} x^3 y \\ \gamma_{xy} &= a_8 + (a_5 + a_9) x + (a_{10} + a_7) y \end{aligned} \tag{15}$$

$$\begin{aligned} u &= a_1 - a_3 y + a_4 x + a_5 xy + 0.5 a_8 y + a_{10} y^2 / 2 + a_{11} x y^2 + a_{12} x^2 y^3 \\ v &= a_2 + a_3 x + a_6 y + a_7 xy + a_8 x / 2 + a_9 x^2 / 2 - a_{11} x^2 y - a_{12} x^3 y^2 \\ \theta_z &= a_3 - 0.5 a_5 x + 0.5 a_7 y + 0.5 a_9 x - 0.5 a_{10} y - 2 a_{11} xy - 3 a_{12} x^2 y^2 \end{aligned} \tag{16}$$

With the same previous procedure of calculation, the obtained shape functions matrix [N] and the strain matrix [B] of SBRIEIR are given consequently in Table 3 and table 4.

Table 3. Shape functions of SBRIEIR element

$N_1^u = 1/4(0.25\xi^2\eta^3 + \zeta\eta - \zeta - 1.25\eta + 1)$	$N_1^v = 1/4(-0.25\xi^3\eta^2 + 0.5\zeta^2 + 0.25\zeta - 0.5)$
$N_2^u = 1/4(-0.25\xi^2\eta^3 + 0.5\eta^2 + 0.25\eta - 0.5)$	$N_2^v = 1/4(0.25\zeta^3\eta^2 + \zeta\eta - 1.25\zeta - \eta + 1)$
$N_3^u = 1/4(-0.5\xi^2\eta^3 - 0.5\zeta\eta^2 + 0.5\zeta + \eta^2 + 0.5\eta - 1)$	$N_3^v = 1/4(-0.5\xi^3\eta 2 + 0.5\zeta^2\eta - 0.5\zeta - \zeta^2 - 0.5\eta + 1)$
$N_4^u = 1/4(-0.25\xi^2\eta^3 - \zeta\eta - \zeta + 1.25\eta + 1)$	$N_4^v = 1/4(0.25\xi^3\eta^2 - 0.5\zeta^2 - 0.25\zeta + 0.5)$
$N_5^u = 1/4(-0.25\xi^2\eta^3 - 0.5\eta^2 + 0.25\eta + 0.5)$	$N_5^v = 1/4(0.25\zeta^3\eta^2 - \zeta\eta - 1.25\zeta + \eta + 1)$
$N_6^u = 1/4(-0.5\xi^2\eta^3 + 0.5\zeta\eta^2 - 0.5\zeta - \eta^2 + 0.5\eta + 1)$	$N_6^v = 1/4(0.5\xi^3\eta 2 - 0.5\zeta^2\eta - 0.5\zeta - \zeta^2 + 0.5\eta + 1)$
$N_7^u = 1/4(-0.25\xi^2\eta^3 + \zeta\eta + \zeta + 1.25\eta + 1)$	$N_7^v = 1/4(0.25\xi^3\eta^2 + 0.5\zeta^2 - 0.25\zeta - 0.5)$
$N_8^u = 1/4(0.25\xi^2\eta^3 + 0.5\eta^2 - 0.25\eta - 0.5)$	$N_8^v = 1/4(-0.25\zeta^3\eta^2 + \zeta\eta + 1.25\zeta + \eta + 1)$
$N_9^u = 1/4(-0.5\xi^2\eta^3 - 0.5\zeta\eta^2 + 0.5\zeta - \eta^2 + 0.5\eta + 1)$	$N_9^v = 1/4(0.5\xi^3\eta 2 + 0.5\zeta^2\eta - 0.5\zeta + \zeta^2 - 0.5\eta - 1)$
$N_{10}^u = 1/4(-0.25\xi^2\eta^3 - \zeta\eta + \zeta - 1.25\eta + 1)$	$N_{10}^v = 1/4(-0.25\xi^3\eta^2 - 0.5\zeta^2 + 0.25\zeta + 0.5)$
$N_{11}^u = 1/4(0.25\xi^2\eta^3 - 0.5\eta^2 - 0.25\eta + 0.5)$	$N_{11}^v = 1/4(-0.25\zeta^3\eta^2 - \zeta\eta + 1.25\zeta - \eta + 1)$
$N_{12}^u = 1/4(-0.5\xi^2\eta^3 + 0.5\zeta\eta^2 - 0.5\zeta + \eta^2 + 0.5\eta - 1)$	$N_{12}^v = 1/4(0.5\xi^3\eta 2 - 0.5\zeta^2\eta - 0.5\zeta + \zeta^2 + 0.5\eta - 1)$

Table 4. The components of the strain matrix [B]of the SBRIEIR

$B_{11} = N_{1,x}^u$	$B_{21} = N_{1,y}^v$	$B_{31} = N_{1,y}^u + N_{1,x}^v$
$B_{12} = N_{2,x}^u$	$B_{22} = N_{2,y}^v$	$B_{32} = N_{2,y}^u + N_{2,x}^v$
$B_{13} = N_{3,x}^u$	$B_{23} = N_{3,y}^v$	$B_{33} = N_{3,y}^u + N_{3,x}^v$
$B_{14} = N_{4,x}^u$	$B_{24} = N_{4,y}^v$	$B_{34} = N_{4,y}^u + N_{4,x}^v$
$B_{15} = N_{5,x}^u$	$B_{25} = N_{5,y}^v$	$B_{35} = N_{5,y}^u + N_{5,x}^v$
$B_{16} = N_{6,x}^u$	$B_{26} = N_{6,y}^v$	$B_{36} = N_{6,y}^u + N_{6,x}^v$
$B_{17} = N_{7,x}^u$	$B_{27} = N_{7,y}^v$	$B_{37} = N_{7,y}^u + N_{7,x}^v$
$B_{18} = N_{8,x}^u$	$B_{28} = N_{8,y}^v$	$B_{38} = N_{8,y}^u + N_{8,x}^v$
$B_{19} = N_{9,x}^u$	$B_{29} = N_{9,y}^v$	$B_{39} = N_{9,y}^u + N_{9,x}^v$
$B_{110} = N_{10,x}^u$	$B_{210} = N_{10,y}^v$	$B_{310} = N_{10,y}^u + N_{10,x}^v$
$B_{111} = N_{11,x}^u$	$B_{211} = N_{11,y}^v$	$B_{311} = N_{11,y}^u + N_{11,x}^v$
$B_{112} = N_{12,x}^u$	$B_{212} = N_{12,y}^v$	$B_{312} = N_{12,y}^u + N_{12,x}^v$

2. Linear numerical validation

2.1. Mac-Neal's elongated cantilever beam

The problem of a cantilever beam shown in Figure 3 has been treated by Mac-Neal and Harder [16]. The beam is subjected to a concentrated force shearing at the free end (P=1) and to a pure bending moment (M=10). It has Young's modulus E $=10^7$, Poison's ratio v=0.3, and a thickness t=0,1.

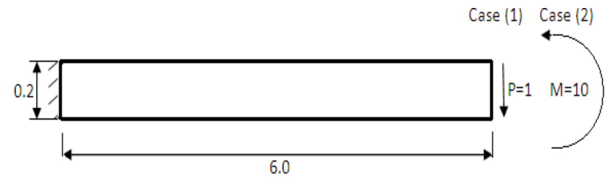

Figure. 3 Mac-Neal's elongated beam subject to (1) end shear and (2) end bending

The results of the normalized deflection at the free end presented in Tables 5 and 6 show that:

- The isoparametric elements SBRIEIR and SBRIE give the same results than the elements SBRIEIR [15] and SBRIE [14] in both cases.

Table 5. Normalized deflection Mac-Neal's elongated beam subjected to end shear

Load case (1): Force shearing at the free end P=1				
Mesh	Isop SBRIEIR	SBRIEIR	Isop SBRIE	SBRIE
6 x 1 12x1	0.9035 0.9083	0.9035 0.9083	0.9035 0.9083	0.9035 0.9083
Reference solution[17]	1,000 (0.1081)			

Table 6. Normalized deflection Mac-Neal's elongated beam subjected to end pure bending

Load case (2): Pure bending moment M=10				
Mesh	Isop SBRIEIR	SBRIEIR	Isop SBRIE	ISBRIE
6 x 1 12x1	0.910 0.910	0.910 0.910	0.910 0.910	0.910 0.910
Reference solution [17]	1,000 (0.270)			

2.2. Plane flexure of cantilever beam

The objective of this problem is to calculate the deflection V_A at the free end of a cantilever beam, with uniform cross-section, subjected to uniform vertical load with Young's modulus $E=10^7$, Poison's ratio v=0.3 as shown in Figure 4.

This problem has been treated by Batoz in [18]. Table 3 shows the results obtained for different mesh for this problem.

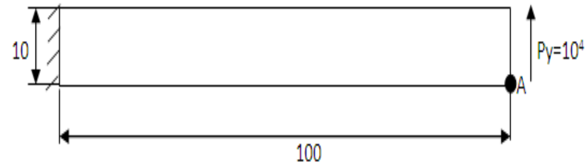

Figure. 4 Cantilever beam subjected to uniform vertical load

Table 7. Displacement V_A of the beam in plane flexure

Elements				
Mesh	Iso SBRIEIR	SBRIEIR	Iso SBRIE	SBRIE
1 x 1	2.75* (12)**	2.75 (12)	2.76 (12)	2.75 (12)
2 x 1	3.43 (18)	3.43 (18)	3.44 (18)	3.43 (18)
3 x 1	3.56 (24)	3.56 (24)	3.56 (24)	3.56 (24)
Beam theory [17] V_A= 4,03 *V_A vertical displacement in A; EI: exact integration; AI: analytical integration; HP hammer point; ** TNDF: Total number of degree of freedom				

The results presented in Table 3 show that the two elements gives the same results for all mesh.

3. Elasto-plastic analysis

In this study two different yield criteria are employed which are; the Von Mises, and the Mohr Coulomb criterion. The constant stiffness method is adopted for the sake of its simplicity, which involves constant stiffness iterations in which non-linearity is introduced by iteravely modifying the right hand side loads vector. The usually elastic global stiffness matrix in such an analysis is formed ones only. Convergence is said to occur when stresses generated by the loads satisfy some yield or failure criterion within prescribed tolerances. The loads vector at each iteration consists of externally applied loads and self equilibrating body-loads. This analysis employs two methods for generating body-loads: visco-plastic and initial stress method to predict the response to loading of an elastic perfectly plastic material. All these methods and yield criteria are given in [19,20].

The aim of this study is to show the performance of the two elements SBRIE SBRIEIR compared to the 8-Node quadrilateral element, to the analytical solutions in elasto-plastic analysis. Two numerical problems are presented; in each problem reduced integration is used for Gaussian quadrature.

3.1. Bearing capacity analysis of purely coherent soil

The elastic properties, Young's modulus, Poisson's ratio, the undrained cohesion and the uniform stress as consistent with [20], were chosen as $E=10^5$ kN/m^2, ν=0.3 , Cu=100 kN/m^2, and q=1kN/m^2 respectively. The Figure 5 shows the geometrical characteristics and meshing of the flexible strip footing. Bearing failure in this problem occurs when q reaches the Prandtl load given by:

$$q_{ultime} = (2+\pi)Cu \tag{17}$$

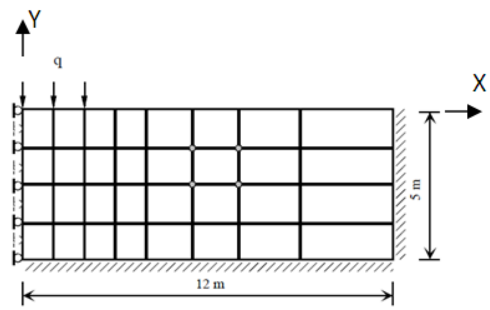

Figure 5. Geometry and mesh of the flexible strip footing

As shown in figure 6 the results, found by the elements (SBRIE and SBRIEIR) have been plotted in the form of a dimensionless bearing capacity factor q/cu versus centerline displacement. These Results shows that these two finite elements have similar results than the Q8 element but the later element uses more degrees of freedom

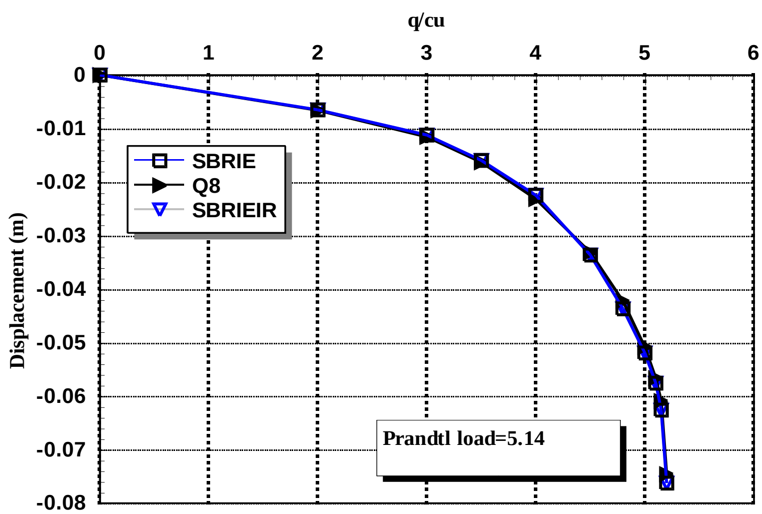

Figure 6. Bearing stress versus centerline displacement

3.2. Stability of a slope subjected to gravity loading

In order to check the accuracy of the present elements SBRIE and SBRIEIR, in this example (figure 5), the geometrical characteristics, material properties, criterion and conditions were chosen as the same of those used in [20]. The factor of safety (F) of the slope is to be assessed, and this quantity is defined as the proportion by which tang φ (friction angle) and Cohesion C must be reduced in order to cause failure. The gravity loading vector is given by

$$P_a = \gamma \sum \iint N^T dxdy \tag{18}$$

Where γ is the unit weight of the material (γ=20kN/m^3), and N is the shape functions.

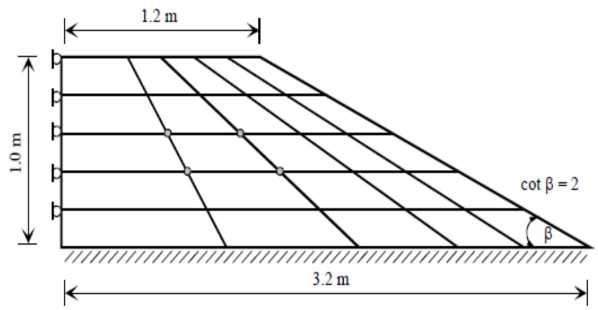

Figure 5. Slope subjected to gravity loading

Results presented in figure 6 in terms of the factor of safety and the maximum of displacement at convergence show that the convergence to the reference solution given in [20] with both elements (SBRIE and SBRIEIR) is quite rapid and similar to the Q8 element.

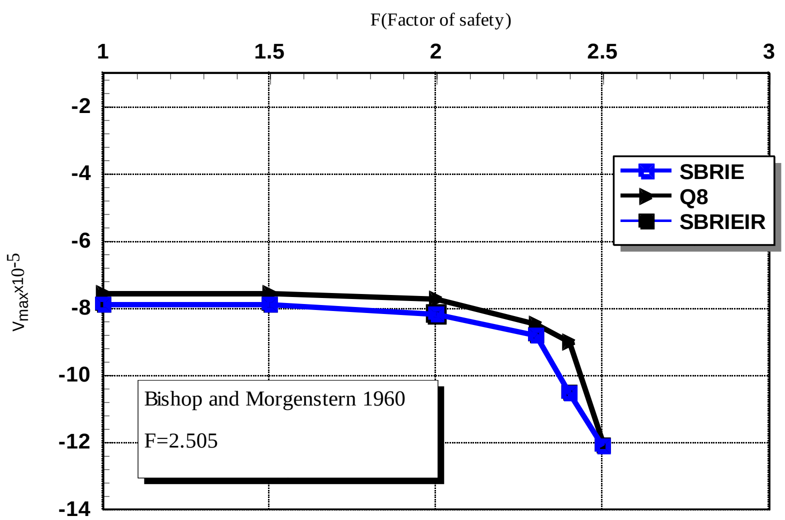

Figure 6. Maximum of displacement versus Factor of safety

5. Conclusions

Two finite elements based on the strain approach named SBRIE and SBRIEIR have been reformulated using the isoparametric procedure and are used in the elastic and elasto-plastic analysis. These elements are simples and contain higher order of polynomial functions. Numerical results obtained in both analyses, agree well with those from the theoretical solutions and show that these elements have the similar behavior than the Q8 element in elastoplastic analysis but they can be less expensive. These elements have quite rapid rate of convergence to the reference solutions for all tests, their performances have been demonstrated, and the advantages of using the strain approach in elasto-plastic analysis have been confirmed.

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